Derivation of the Intriligator-Leigh-Seiberg linearity principle from Dijkgraaf-Vafa

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1. Introduction

the Dijkgraaf-Vafa proposal

Duality between
$d=4, \mathcal{N}=1$ SYM theories
and (old) matrix models:

- **Gauge Theory** : N_c Finite and Fixed
  the bare superpotential $W_{\text{tree}}(\Phi)$
  $\Downarrow$
  the effective superpotential $W_{\text{eff}}(S)$
  (Φ: adjoint matter, $N_c \times N_c$
  S: gaugino condensate $(1/32\pi^2)\,\text{tr}\, W^\alpha W_\alpha$)

- **Matrix Model** : Planar Limit
  the action $W_{\text{tree}}(\Phi)$
  $\Downarrow$
  the free energy $\mathcal{F}(S)$
  (Φ: $M \times M$ matrix, S: 't Hooft coupling)
  $M \to \infty$ with S fixed

- The correspondence is

$$W_{\text{eff}}(S) = N_c \frac{\partial \mathcal{F}}{\partial S}$$

and it was conjectured to be EXACT

First Indication — Factorization

⋆ Matrix Model Side

• Consider a diagram with V vertices, E propagators, h index loops with Euler number χ

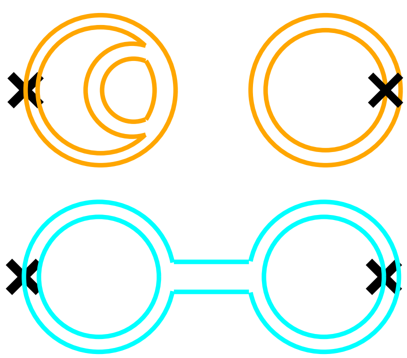
$$V - E + h = \chi$$

• $(\text{Propagators})^{-1}$ and couplings $\propto M/S$

$$\Downarrow$$

$$\text{diagram} \propto M^h \left(\frac{M}{S}\right)^{V-E} = M^{\chi} S^{V-E}$$

In the planar limit $M \to \infty$, planar diagrams dominate and correlation functions factorize:($E = 5$, $V = 4$)



$h = 5 \rightsquigarrow M^4 S^2$

$h = 3 \rightsquigarrow M^2 S^2$

⋆Gauge theory side

VEVs of the lowest components of gauge invariant chiral superfields factorize:

$$\langle \Phi_1(x)\Phi_2(y)\rangle = \langle \Phi_1(x)\rangle\langle \Phi_2(y)\rangle$$

It is because

- $\partial_\mu \Phi_1(x) = \{\bar{D}, [D, \Phi_1(x)]\}$ **is $\bar{D}$-exact.**
- **Hence,**

$$\begin{aligned}\partial_\mu \langle \Phi_1(x)\Phi_2(y)\rangle &= \langle \{\bar{D}, [D, \Phi_1(x)]\}\Phi_2(y)\rangle \\ &= \langle [D, \Phi_1(x)]\{\bar{D}, \Phi_2(y)\}\rangle \\ &= 0\end{aligned}$$

- **Next, Take $|x-y| \to \infty$**
- **Finally use the Cluster Property.**

Stringy ideas behind the proposal

DV arrived this proposal by chasing the following dualities:

$\mathcal{N}=1$ SYM	!!! $\longleftrightarrow$	**matrix model**
$\cap$		$\Vert$
D5 wrapped on S^2 in CY	twist $\Longrightarrow$	**B brane on S^2 in CY**
large N dual $\updownarrow$		$\updownarrow$ large N dual
IIB on CY with flux in S^3	twist $\Longrightarrow$	**B model on CY with flux in S^3**

Checking the proposal

To calculate $W_{\text{eff}}(S)$, one usually combines

- exact results (the Seiberg-Witten curve or the Affleck-Dine-Seiberg superpotential), which describe the system without the deformation $W_{\text{tree}}(\Phi)$.
- the linearity principle of Intriligator-Leigh-Seiberg, which tells the response of the system to $W_{\text{tree}}(\Phi)$.

(more on this in section 2.)

Extending to the fundamental flavors

$$W_{\text{eff}}(S) = N_c \frac{\partial \mathcal{F}_{\text{sphere}}}{\partial S} + \mathcal{F}_{\text{disk}}.$$

It is now known that diagrams with more than two boundaries do not contribute.

Field-theoretic derivations of the proposal

By now, we have two derivations of the conjecture:

- Dijkgraaf-Grisaru-Lam-Vafa-Zanon

By purterbatively integrating out the matter superfields in an external gaugino condensate
(more on this in section 3.)

- Cachazo-Douglas-Seiberg-Witten

By combining the generalized Konishi anomaly relations and factorization of vevs of chiral superfields

$$\Downarrow$$

We can study the dynamics of $\mathcal{N} = 1$ SYM systems using DV!
As an example, we showed from DV the ILS linearity principle .

2. Brief Review of ILS

From the bare lagrangian

$$\mathcal{L}_{\text{bare}} = \int d^4\theta \Phi^\dagger e^V \Phi + \int d^2\theta W(\Phi, g_i) + c.c.$$

$$\Downarrow$$

To the low energy superpotential

$$\int d^2\theta \; W_{\text{eff}}(\Phi)$$

holomorphy

Coupling constants g_i as the vevs of chiral superfields

$$\rightsquigarrow$$

$W_{\text{eff}}(\Phi)$ depends only on g_i , not on $\overline{g_i}$.

symmetry

• Many classical symmetries on chiral superfields and coupling constants.

• Even anomalous symmetries are useful once one assigns charges to the dynamical scale Λ appropriately.

classical and perturbative limit

- In the classical limit $\Lambda \to 0$, $W_{\text{eff}}(\Phi)$ must approach $W_{\text{tree}}(\Phi)$.
- If one takes some coupling $g_i \to 0$, W_{eff} should smoothly become W_{tree}.

$$\Downarrow$$

W_{eff} does not contain negative powers of Λ and g_i.

Strong constraint for possible terms in $W_{\text{eff}}(\Phi)$,
fixing the form completely in some simple cases .

Example.

$SU(N_c)$ SYM with $N_f < N_c$ pairs of quarks Q_i and $\tilde{Q}_i$.

- Gauge invariants are

$$T_{ij} = Q_i \tilde{Q}_j.$$

- $W_{\text{tree}} = m_{ij} T_{ij}$ as the bare superpotential
- $SU(N_f) \times SU(N_f)$ flavor symmetry on quarks and antiquarks.

- R-charges:

	θ	Q	$\tilde{Q}$	m	W_α
R-charge	1	0	0	2	1

- The anomaly from

$$q \to e^{-i\phi} q, \qquad \lambda \to e^{+i\phi}\lambda,$$

leads to:

$$\theta \to \theta + 2(N_c - N_f)\phi.$$

Because

$$\Lambda^{3N_c - N_f} \sim \Lambda_0^{3N_c - N_f} \exp\left(-\frac{8\pi^2}{g_0^2} + i\theta\right),$$

the R-charge of $\Lambda^{3N_c - N_f}$ is $2(N_c - N_f)$.

$$\Downarrow$$

$$W_{\text{eff}} = c\left(\frac{\Lambda^{3N_c - N_f}}{\det T_{ij}}\right)^{1/(N_c - N_f)} + \underbrace{m_{ij} T_{ij}},$$

the coeff. is fixed by $m \to 0$ limit.

the linearity principle

- The result can be summarized as

$$W_{\text{eff}} = \underbrace{W_{\text{n.p.}}}_{\substack{\text{non-perturbative,} \\ \text{independent of} \\ \text{coupling constants}}} + W_{\text{tree}}$$

- Various other models have this form. ILS proposed this linearity as an organizing principle.

- However, symmetry arguments are not always strong enough.

Another Example

- Consider $\mathcal{N} = 1$ $SU(N_c)$ SYM with one adjoint Φ.

- The R-charge of Λ is zero .

- Introduce the bare superpotential

$$m \, \text{tr} \, \Phi^2/2.$$

- Symmetry alone cannot exclude the corrections of the form $m\Lambda^2$.

3. Brief Review of DGLVZ

Strategy

1. Introduce an external gauge superfield. In particular, introduce a space-time constant gaugino.
2. Perturbatively integrate out the matter superfields. $W_{\text{pert}}(S)$
3. Make the gauge superfield dynamical.

$$W_{\text{eff}}(S) = \underbrace{W_{\text{VY}}(S)}_{\substack{\text{dynamical effect} \\ \text{from the gauge superfield}}} + W_{\text{pert}}(S)$$

the propagator

$$\langle \Phi\Phi \rangle = \frac{\bar{m}}{p^2 + m\bar{m} + W^\alpha \pi_\alpha}$$

where

p	:	bosonic momentum,
W^α	:	field strength superfield,
π_α	:	fermionic momentum.

Integration of loop momenta:

$$\prod_{a=1}^{l} \left(\int \frac{d^4 p_a}{(2\pi)^4} \int d^2 \pi_a \right)$$

$\Longrightarrow$ Every l-loop diagram carries a factor of $\; {W_\alpha}^{2l}$.

$\Longrightarrow$ if $W_\alpha = 0$, no correction to F terms.

This is the Perturbative Non-Renormalization Theorem!

What if $W_\alpha \neq 0$?

Origin of Planarity

- $\mathrm{tr}\, W_\alpha^m$ is $\bar{D}$-exact for $m \geq 3$, i.e. $\sim$ zero in the F terms. (CDSW)

- Therefore, W_{eff} is a function of

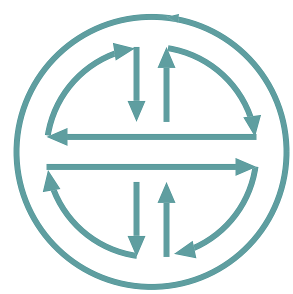
$$w_\alpha = \mathrm{tr}\, W_\alpha \quad \text{and} \quad 32\pi S = \mathrm{tr}\, W^\alpha W_\alpha.$$

To give non-zero requires at least l index loops.

- \# of index loops h satisfies

$$h = l + 1 - 2 \times \text{genus}.$$

$l = 3$ $\quad\wedge\quad$ $h = 4$



$l = 3$ $\quad\vee\quad$ $h = 2$

Only GENUS 0 diagrams contribute!

- Further tricks reveal the resulting Feynman rules are the same as those of the matrix models in the planar limit.

Making the vector superfield dynamical

This introduces the Veneziano-Yankielowicz term

$$W_{\text{VY}} = N_c S(1 - \log \frac{S}{\Lambda^3}).$$

(Λ: the dynamical scale of SQCD)

There are arguments against further corrections.

The total effective superpotential:

$$W_{\text{eff}}(S) = W_{\text{VY}}(S) + W_{\text{pert}}(S).$$

- S is determined by the extremalization

$$\partial W_{\text{eff}} / \partial S = 0.$$

- For $W_{\text{tree}} = \sum g_i \mathcal{O}_i$, the vevs are

$$\begin{aligned} \langle \mathcal{O}_i \rangle &= \frac{\partial}{\partial g_i} W_{\text{eff}}(S(g_i), g_i) \\ &= \left. \frac{\partial W_{\text{eff}}}{\partial g_i} \right|_S + \frac{\partial S}{\partial g_i} \left. \frac{\partial W_{\text{eff}}}{\partial S} \right|_{g_i} = \left. \frac{\partial W_{\text{eff}}}{\partial g_i} \right|_S . \end{aligned}$$

4. ILS from DGLVZ

• Introduce $W_{\text{tree}} = \sum g_i \mathcal{O}_i$.

• From DV, calculate W_{eff} and $\langle W_{\text{tree}} \rangle$ in $W_{\text{eff}} = W_{\text{n.p.}} + \langle W_{\text{tree}} \rangle$.

• Is $W_{\text{n.p.}}$ independent of g_i as a function of Λ , g_i and $\langle \mathcal{O}_i \rangle$? This is the linearity principle.

Restriction on the matter rep.

We impose:

• Vector-like , i.e. gauge-invariant mass terms can be given to all fields.

• No $U(1)$ factor left in low energy.

• Each gauge invariants $\mathcal{O}_i$ is a polynomial of F_a's.

• Either the R-charge of Λ vanishes, or satisfies the Restriction ($\heartsuit$) :

no dynamical constraints among $\langle F_a \rangle$

$$P(\langle F_1 \rangle, \langle F_2 \rangle, \cdots) = 0.$$

cf.

(♡) may sounds a serious restriction, but please note:

Take for example $Sp(N)$ SYM with $2N_f$ fundamentals Q_i.

$N+1>N_f$: No constraints.

$N+1=N_f$: A dynamical constraint

$$\text{Pf}\langle Q_i Q_j\rangle = \Lambda^{2N_f},$$

but in this case the R-charge of Λ is zero.

$N+1<N_f$: Classical constraints

$$\text{Pf}\langle Q_i Q_j\rangle = 0,$$

but all lifted dynamically.

- All satisfy the restriction we imposed.
- The $SU(N_f)$ case is quite similar.

Step 1.

We show $W_{\text{n.p.}} = (N_c - N_f)S$.

- $\underbrace{V}_{\#(\text{vertices})} - \underbrace{E}_{\#(\text{propagators})} + \underbrace{L}_{\#(\text{loops})} = 1$

- V, E and L can be counted by

$$g_j\partial/\partial g_j, \quad -m_i\partial/\partial m_i, \quad S\partial/\partial S$$

- Hence for each diagram D,

$$\left(1 - S\frac{\partial}{\partial S}\right) D = \underbrace{\left(\sum m_i \frac{\partial}{\partial m_i} + \sum g_j \frac{\partial}{\partial g_j}\right) D}_{\text{the contrib. of } D \text{ to } \langle W_{\text{tree}}\rangle},$$

i.e.

$$\langle W_{\text{tree}}\rangle = \left(1 - S\frac{\partial}{\partial S}\right)\sum D$$

modulo subtlety at one-loop.

(YT, hep-th/0211189)

- Because $\dfrac{\partial W_{\text{eff}}}{\partial S} = \dfrac{\partial}{\partial S}(W_{\text{VY}} + \sum D) = 0$,

$$\langle W_{\text{tree}}\rangle = \sum D + S\frac{\partial W_{\text{VY}}}{\partial S}.$$

- Hence $W_{\text{n.p.}} = (N_c - N_f)S$.

Note $N_c - N_f$ is half the R-charge of Λ^β

Step 2.

We show S is **independent of g_i** when expressed as a function of Λ and $\langle O_i \rangle$.

$$\Uparrow$$

The same as inquiring whether S does **not change** when g_i's are varied as long as $\underbrace{\langle O_i \rangle \text{ remain fixed}}_{\text{by the factorization, the same as fixing } \langle F_a \rangle}$.

<DERIVATION>

- From $\delta(\partial W_{\text{eff}}/\partial S) = 0$,

$$-\frac{\partial^2 W_{\text{eff}}}{\partial S \partial S}\delta S = \sum_i \delta\lambda_i \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial S}.$$

- Hence $\delta\langle F_j \rangle = \sum_i \delta\lambda_i G_{ij}$ where

$$G_{ij} = \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial \lambda_j} - \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial S}\frac{\partial^2 W_{\text{eff}}}{\partial S \partial \lambda_j}\Big/\frac{\partial^2 W_{\text{eff}}}{\partial S \partial S}$$

$$= \frac{\partial \langle O_i \rangle}{\partial \lambda_j} - \frac{\partial \langle O_i \rangle}{\partial S}\frac{\partial^2 W_{\text{eff}}}{\partial S \partial \lambda_j}\Big/\frac{\partial^2 W_{\text{eff}}}{\partial S \partial S}.$$

• From factorization,

$$G_{ij} = \langle \frac{\partial O_i}{\partial F_a} \rangle G_{aj}.$$

Hence

$$\delta \langle F_j \rangle = \left(\delta \lambda_i \langle \frac{\partial O_i}{\partial F_a} \rangle \right) G_{aj}$$

• Similarly, one obtains

$$-\frac{\partial^2 W_{\text{eff}}}{\partial S \partial S} \delta S = \left(\delta \lambda_i \langle \frac{\partial O_i}{\partial F_a} \rangle \right) \frac{\partial \langle F_a \rangle}{\partial S}$$

• The restriction ($\heartsuit$) ensures that the rank of G_{ij} is maximal.

Thus, $\delta \langle F_i \rangle = 0$ implies $\delta S = 0$.

Result

• If the R-charge of Λ vanishes, $W_{\mathrm{n.p.}} = 0$ from Step 1.

• If it does not vanish, by the restriction ($\heartsuit$) and Step 2, $W_{\mathrm{n.p.}} \propto S$ is independent of the coupling constants.

This is the linearity principle.   Q.E.D.

N.B.

When the rank of G_{ij} is not maximal, G_{ij} contains the info of the dynamical constraints. Further, G_{ij} can be computed perturbatively!

$$\Downarrow$$

The dynamical constraints, if any, can be readily studied in this approach.

5. Conclusions

Summary

• We derived the ILS linearity principle in the framework of Dijkgraaf-Vafa

Future work

Directly related to this work:

• Lifting the restriction (♡)

More generally, interesting directions around the DV framework are:

• Direct topological twisting of $\mathcal{N} = 1$ SYM theories

• Extending to matter superfields not in a vector-like representation.